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Hamiltonian and BRST Formulations of the Nielsen–Olesen Model

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The Hamiltonian and BRST formulations of the Nielsen–Olesen (vortex) model are studied in two space, one time dimensions.

1. INTRODUCTION

The systems in two space, one time ((2+1)-) dimensions, i.e., the planar systems, display a variety of peculiar quantum mechanical phenomena ranging from massive gauge fields to soluble gravity (Forte, 1992; Jackiw, 1989; Krive and Rozhavskii, 1987; Saint-James *et al.*, 1969). These are linked to the peculiar structure of the rotation, Lorentz and Poincare groups in (2+1)-dimensions. The (2+1)-dimensional QED models with a Higgs potential, namely, the abelian Higgs models involving the vector guage field $A^{\mu}(x)$ with and without the topological Chern–Simons term in two space, one time ((2+1)-) dimensions, have been of a wide interest in the recent years (Abrikosov, 1957a,b; Banerjee *et al.*, 1995, 1997; Banks and Lykken, 1990; Bogomol'nyi, 1976a,b; Chen *et al.*, 1989; Daser *et al.*, 1982a,b; De Vega and Schaposnik, 1976; Dunne and Trugenberger, 1991; Fetter *et al.*, 1989; Forte, 1992; Friedberg and Lee, 1977a,b, 1978; Ginsburg and Landau, 1950; Jackiw, 1989; Jackobs and Rebbi, 1986; Krive and Rozhavskii, 1987; Laughlin, 1988; Lee *et al.*, 1991; Lee and Nam, 1991; Mac Kenzie and Wilczek, 1988; Nielsen and Olesen, 1973a,b; Saint-James *et al.*, 1969; Shin *et al.*, 1990). Such models when considered with a CST in the action may be considered as field-theoretical models for anyons (Banerjee *et al.*, 1995, 1997; Daser *et al.*, 1982a,b; Dunne and Trugenberger, 1991; Forte, 1992; Jackiw, 1989; Krive and Rozhavskii, 1987; Mac Kenzie and Wilczek, 1988; Laughlin, 1988; Saint-James *et al.*, 1969; Shin *et al.*, 1990). When these models are considered without a CST

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in the action but only with a Maxwell term accounting for the kinetic energy of the vector gauge field $A^{\mu}(x)$ (Abrikosov, 1957a,b; Banks and Lykken, 1990; Bogomol'nyi, 1976a,b; Chen *et al.*, 1989; De Vega and Schaposnik, 1976; Fetter *et al.*, 1989; Forte, 1992; Friedberg and Lee, 1977a,b, 1978; Ginsburg and Landau, 1950; Jackiw, 1989; Jackobs and Rebbi, 1986; Krive and Rozhavskii, 1987; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b; Saint-James *et al.*, 1969), they represent field-theoretical models which could be considered as effective theories of the Ginsburg–Landau-type for superconductivity (Abrikosov, 1957a,b; Banks and Lykken, 1990; Chen *et al.*, 1989; Fetter *et al.*, 1989; Ginsburg and Landau, 1950). These models in (2+1)- or (3+1)-dimensions are known as the Nielsen–Olesen (vortex) models (Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Friedberg and Lee, 1977a,b, 1978; Jackobs and Rebbi, 1986; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b), which are in fact the relativistic generalizations of the well-known Ginsburg–Landau phenomenological field-theory models of superconductivity (Abrikosov, 1957a,b; Ginsburg and Landau, 1950). Some basics of the Nielsen–Olesen (vortex) model are recapitulated in the next section (Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Friedberg and Lee, 1977a,b, 1978; Jackobs and Rebbi, 1986; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b).

Also, the quantization of field-theory models has always been a challenging problem. Infact, any complete physical theory is a quantum theory and the only way of defining a quantum theory is to start with a classical theory and then to quantize it. Basically there are two rather equivalent approaches for the quantization: the canonical quantization or the Hamiltonian formulation due to Dirac (1950, 1964) and the path integral quantization due to Feynman. In the present work we consider a consistent Hamiltonian (Dirac, 1950, 1964) and Becchi–Rouet–Stora–Tyutin (BRST) (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin) quantization of the Nielsen–Olesen model in (2+1)-dimensions with some specific gauge choices (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin).

Further, in the usual Hamiltonian formulation of a gauge-invariant theory under some gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under the gauge fixing). To achieve the quantization of a gauge-invariant theory such that the gauge invariance of the theory is maintained even under gauge fixing, one goes to a more generalized procedure called the BRST formulation (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin). In the BRST formulation of a gauge-invariant theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the gauge-invariant theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators having different statistics. In view of this, one introduces new anticommuting variables c and $\bar{c}$ called the Faddeev–Popov ghost and antighost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi–Lautrup field (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin). In the BRST formulation, one thus embeds a gauge-invariant theory into a BRST-invariant system, and the quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator Q as well as anti-BRST charge operator $\bar{Q}$. The new symmetry of the quantum system (the BRST symmetry) that replaces the gauge invariance is maintained (even under the gauge fixing) and hence projecting any state onto the sector of BRST and anti-BRST invariant state yields a theory that is isomorphic to the original gauge-invariant theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian is guaranteed by the conservation and nilpotency of the BRST charge (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin).

After a brief recapitulation of the basics of the model in the next section, its Hamiltonian formulation is considered in Section 3 and its BRST formulation is studied in Section 4.

2. A RECAPITULATION OF SOME BASICS OF THE NIELSEN–OLESEN MODEL

The Nielsen–Olesen model in two-space, one-time dimensions is an abelian Higgs model defined by the action (Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Friedberg and Lee, 1977a,b, 1978; Jackobs and Rebbi, 1986; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b)

$$S = \int \mathscr{L}(\Phi, \Phi^*, A^\mu)\, d^3x \tag{2.1a}$$

$$\mathscr{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + (\tilde{D}_\mu \Phi^*)(D^\mu \Phi) - V(|\Phi|^2) \tag{2.1b}$$

$$V(|\Phi|^2) = \alpha_0 + \alpha_2 |\Phi|^2 + \alpha_4 |\Phi|^4 \tag{2.1c}$$

$$= \lambda\left(|\Phi|^2 - \Phi_0^2\right)^2, \quad \Phi_0 \neq 0 \tag{2.1d}$$

$$D_\mu = (\partial_\mu + ieA_\mu), \qquad \tilde{D}_\mu = (\partial_\mu - ieA_\mu) \tag{2.1e}$$

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2.1f}$$

$$g^{\mu\nu} := \text{diag}(+1, -1, -1), \quad \mu, \nu = 0, 1, 2 \tag{2.1g}$$

This model defined in (2+1)-dimensions as well as in (3+1)-dimensions is widely known as the Nielsen–Olesen (vortex) model. In the present work, however, we would study this model in (2+1)-dimensions only. This model is in fact a relativistic generalization of the well-known Ginsburg–Landau model, which is a phenomenological field-theory model of superconductivity (Abrikosov, 1957a,b; Ginsburg and Landau, 1950). The model is well known to possess stable, time-independent (i.e., static), classical solutions called as the two-dimensional solitons, which are in fact the topological solitons of the vortex type (Banerjee *et al.*, 1995, 1997; Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Dunne and Trugenberger, 1991; Forte, 1992; Friedberg and Lee, 1977a,b, 1978; Jackiw, 1989; Jackobs and Rebbi, 1986; Krive and Rozhavskii, 1987; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b; Saint-James *et al.*, 1969).

In a quantum theory of the kind that we are considering here, for a specific form of the Higgs potential which admits static solutions, in general, one could have *two* degenerate minima—a symmetry breaking minimum and a symmetry preserving minimum—and correspondingly the theory could have two types of classical solutions—topological vortices with quantized magnetic flux as we have in the Nielsen–Olesen model or Ginsburg–Landau model, where it is possible to define a conserved topological current and a corresponding topological charge which is quantized and is related to the topological quantum number called the winding number, which determines the lower bound of the energy of the vortex solutions (Banerjee *et al.*, 1995, 1997; Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Dunne and Trugenberger, 1991; Friedberg and Lee, 1977a,b, 1978; Jackobs and Rebbi, 1986; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b), and the other type of classical solutions are the nontopological solitons with nonvanishing but not necessarily quantized magnetic flux (Banerjee *et al.*, 1995, 1997; Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Dunne and Trugenberger, 1991; Friedberg and Lee, 1977a,b, 1978; Jackobs and Rebbi, 1986; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b).

In the Nielsen–Olesen model the field $F_{\mu\nu}$ has a simple meaning, namely, the field F_{12} measures the number of vortex lines (going in the third direction) which pass a unit square in the (12)-plane. The vortex line is identified with a dual string and the flux of vortex lines is quantized (with the quantum $(-2\pi/e)$) (Banerjee *et al.*, 1995, 1997; Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Dunne and Trugenberger, 1991; Friedberg and Lee, 1977a,b, 1978; Jackobs and Rebbi, 1986; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b). The main new result of this theory is the identification of the Ginsburg–Landau theory with the static solution of the Higgs type of Lagrangian (Abrikosov, 1957a,b; Banerjee *et al.*, 1995, 1997; Banks and Lykken, 1990; Bogomol'nyi, 1976a,b; Chen *et al.*, 1989; De Vega and Schaposnik, 1976; Dunne and Trugenberger, 1991; Fetter *et al.*, 1989; Forte, 1992; Friedberg and Lee, 1977a,b, 1978; Ginsburg and Landau, 1950; Jackiw, 1989; Jackobs and Rebbi, 1986; Krive and Rozhavskii, 1987; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b; Saint-James *et al.*, 1969).

Further, in the Nielsen–Olesen model, considered with a Higgs potential in the form of a double well potential with $\Phi_0 \neq 0$, the spontaneous symmetry breaking takes place owing to the noninvariance of the lowest (ground) state of the system (because $\Phi_0 \neq 0$) under the operation of the local $U(1)$ symmetry. Also the symmetry that is broken is still a symmetry of the system and it is manifested in a manner other than the invariance of the lowest or ground state (Φ_0) of the system. However, no Goldstone boson occurs here and instead the gauge field acquires a mass through some kind of a Higgs mechanism and the symmetry is manifested in the Higgs mode.

Also, the Nielsen–Olesen model with the parameters of the Higgs potential chosen such that the scalar (spin zero) particle and the vector (spin one) particle masses are equal, i.e., if we set the scalar (Higgs boson) and vector (photon) masses to be equal, i.e.,

$$m_{\text{Higgs}} = m_{\text{photon}} = e\Phi_0$$

so that

$$V(|\Phi|^2) := \frac{1}{2}e^2\left(|\Phi|^2 - \Phi_0^2\right)^2$$

then the model reduces to the so-called Bogomol'nyi model (Banerjee *et al.*, 1995, 1997; Bogomol'nyi, 1976a,b; Dunne and Trugenberger, 1991) which describes a system on the boundary between type-I and type-II superconductivity and admits self-dual solitons (Banerjee *et al.*, 1995, 1997; Bogomol'nyi, 1976a,b; Dunne and Trugenberger, 1991).

In our considerations in the present work, we would keep the Higgs potential rather general, i.e., we would not make any specific choice for the parameters of the potential except that they are chosen such that the potential remains a double well potential with $\Phi_0 \neq 0$. For further details we refer to the work of Banerjee *et al.* (1995, 1997), Bogomol'nyi (1976a,b), De Vega and Schaposnik (1976), Dunne and Trugenberger (1991), Friedberg and Lee (1977a,b, 1978), Jackobs and Rebbi (1986), Lee *et al.* (1991), Lee and Nam (1991), Nielsen and Olesen (1973a,b), and references therein.

In the next section, we consider the Hamiltonian formulation of the Nielsen–Olesen model in (2+1)-dimensions.

3. HAMILTONIAN FORMULATION

For considering the Hamiltonian formulation of the Nielsen–Olesen model in the instant-form (i.e., on the hyperplanes x^0 = constant), we first express the action of the theory (2.1) in the component form, which in (2+1)-dimensions reads as (Banerjee *et al.*, 1995, 1997; Bogomol'nyi, 1976a,b; De Vega and Schaposnik, 1976; Dunne and Trugenberger, 1991; Friedberg and Lee, 1977a,b, 1978; Jackobs and Rebbi, 1986; Lee *et al.*, 1991; Lee and Nam, 1991; Nielsen and Olesen, 1973a,b)

$$S = \int \mathscr{L}\, dx_0\, dx_1\, dx_2 \tag{3.1a}$$

$$\begin{aligned}\mathscr{L} = \frac{1}{2e^2}&\left\{(\partial_1 A_0 - \partial_0 A_1)^2 + (\partial_2 A_0 - \partial_0 A_2)^2 - F_{12}^2\right\}\\ &+ \left\{(\partial_0\Phi^*)(\partial_0\Phi) + ie(\partial_0\Phi^*)A_0\Phi - ieA_0\Phi^*(\partial_0\Phi) + e^2A_0^2\Phi^*\Phi\right\}\\ &+ \left\{-(\partial_1\Phi^*)(\partial_1\Phi) - ie(\partial_1\Phi^*)A_1\Phi + ieA_1\Phi^*(\partial_1\Phi) - e^2A_1^2\Phi^*\Phi\right\}\\ &+ \left\{-(\partial_2\Phi^*)(\partial_2\Phi) - ie(\partial_2\Phi^*)A_2\Phi + ieA_2\Phi^*(\partial_2\Phi) - e^2A_2^2\Phi^*\Phi\right\}\\ &- V(|\Phi|^2)\end{aligned} \tag{3.1b}$$

$$V(|\Phi|^2) = \alpha_0 + \alpha_2|\Phi|^2 + \alpha_4|\alpha|^4 \tag{3.1c}$$

$$= \lambda\left(|\Phi|^2 - \Phi_0^2\right)^2, \quad \Phi_0 \neq 0 \tag{3.1d}$$

where all the symbols are defined in (2.1). Equations (3.1) define the theory in the instant-form in (2+1)-dimensions. In the following, we would consider the Hamiltonian formulation of the theory described by the action (3.1). The Euler–Lagrange field equations of motion of the theory obtained from (3.1) are

$$-ie(\partial_\mu\Phi^*)A^\mu + e^2A_\mu A^\mu\Phi^* - \partial_\mu\partial^\mu\Phi^* + ie\partial_\mu(A^\mu\Phi^*) - \frac{\partial V}{\partial\Phi} = 0 \tag{3.2a}$$

$$-ieA_\mu(\partial_\mu\Phi) + e^2A_\mu A^\mu\Phi - \partial_\mu\partial^\mu\Phi - ie\partial_\mu(A^\mu\Phi) - \frac{\partial V}{\partial\Phi^*} = 0 \tag{3.2b}$$

$$ie\Phi^*(\partial_1\Phi) - ie(\partial_1\Phi^*)\Phi - 2e^2A_1\Phi^*\Phi + \frac{1}{e^2}(\partial_0 F_{10} - \partial_2 F_{12}) = 0 \tag{3.2c}$$

$$ie\Phi^*(\partial_2\Phi) - ie(\partial_2\Phi^*)\Phi - 2e^2A_2\Phi^*\Phi + \frac{1}{e^2}(\partial_0 F_{20} - \partial_1 F_{12}) = 0 \tag{3.2d}$$

$$ie(\partial_0\Phi^*)\Phi - ie(\partial_0\Phi)\Phi^* + 2e^2A_0\Phi^*\Phi + \frac{1}{e^2}(\partial_1 F_{01} - \partial_2 F_{02}) = 0 \tag{3.2e}$$

From the above equations (Eqs. (3.2)) it is easy to see that the vector current of the theory (J^μ) is conserved, i.e.,

$$\partial_\mu J^\mu = \partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2 = 0 \tag{3.3}$$

implying that the theory possesses (at the classical level) a vector-gauge symmetry.

The canonical momenta obtained from (3.1) are

$$\Pi := \frac{\partial \mathscr{L}}{\partial(\partial_0 \Phi)} = \partial_0 \Phi^* - ieA_0 \Phi^* \tag{3.4a}$$

$$\Pi^* := \frac{\partial \mathscr{L}}{\partial(\partial_0 \Phi^*)} = \partial_0 \Phi + ieA_0 \Phi \tag{3.4b}$$

$$\Pi^0 := \frac{\partial \mathscr{L}}{\partial(\partial_0 A_0)} = 0 \tag{3.4c}$$

$$E_1 := \Pi^1 = \frac{\partial \mathscr{L}}{\partial(\partial_0 A_1)} = \frac{-1}{e^2}(\partial_1 A_0 - \partial_0 A_1) \tag{3.4d}$$

$$E_2 := \Pi^2 = \frac{\partial \mathscr{L}}{\partial(\partial_0 A_2)} = \frac{-1}{e^2}(\partial_2 A_0 - \partial_0 A_2) \tag{3.4e}$$

Here Π, Π^*, Π^0, $E_1(:=\Pi^1)$, and $E_2(:=\Pi^2)$ are the momenta canonically conjugate, respectively, to Φ, Φ^*, A_0, A_1, and A_2. Equations (3.4) imply that the theory possesses one primary constraint:

$$\chi_1 = \Pi_0 \approx 0 \tag{3.5}$$

The canonical Hamiltonian density corresponding to $\mathscr{L}$(3.1) is

$$\mathcal{H}_c := \Pi(\partial_0 \Phi) + \Pi^*(\partial_0 \Phi^*) + \Pi_0(\partial_0 A_0) + E_1(\partial_0 A_1) + E_2(\partial_0 A_2) - \mathscr{L} \tag{3.6a}$$

$$\begin{aligned} &= \frac{1}{2}e^2\big(E_1^2 + E_2^2\big) + E_1(\partial_1 A_0) + E_2(\partial_2 A_0) + \Pi^* \Pi \\ &\quad + ie(\Pi^* A_0 \Phi^* - \Pi A_0 \Phi) + e^2 \Phi^* \Phi\big(A_1^2 + A_2^2\big) + (\partial_1 \Phi^*)(\partial_1 \Phi) \\ &\quad + (\partial_2 \Phi^*)(\partial_2 \Phi) + ie\{(\partial_1 \Phi^*) A_1 \Phi + (\partial_2 \Phi^*) A_2 \Phi\} - ie\{(A_1 \Phi^*)(\partial_1 \Phi) \\ &\quad + (A_2 \Phi^*)(\partial_2 \Phi)\} + \frac{1}{2e^2} F_{12}^2 = V(|\Phi|^2) \end{aligned} \tag{3.6b}$$

After including the primary constant χ_1 in the canonical Hamiltonian density $\mathcal{H}_c$ (3.6) with the help of the Lagrange multiplier field u, the total Hamiltonian density $\mathcal{H}_T$ could be written as

$$\mathcal{H}_T := \mathcal{H}_c + \Pi_0 u \tag{3.7}$$

The Hamilton's equations obtained from the total Hamiltonian

$$H_{\mathrm{T}} = \int \mathcal{H}_{\mathrm{T}}\, dx_1\, dx_2 \tag{3.8}$$

are

$$\partial_0 \Phi = \frac{\partial H_{\mathrm{T}}}{\partial \Pi} = \Pi^* - ieA_0\Phi \tag{3.9a}$$

$$\begin{aligned}-\partial_0 \Pi = \frac{\partial H_{\mathrm{T}}}{\partial \Phi} =& -ie\Pi A_0 + e^2\big(A_1^2 + A_2^2\big)\Phi^* - \partial_1\partial_1\Phi^* \\ &- \partial_2\partial_2\Phi^* + ie(\partial_1\Phi^*)A_1 + ie(\partial_2\Phi^*)A_2 \\ &+ ie\partial_1(A_1\Phi^*) + ie\partial_2(A_2\Phi^*) + \frac{\partial V}{\partial \Phi}\end{aligned} \tag{3.9b}$$

$$\partial_0 \Phi^* = \frac{\partial H_{\mathrm{T}}}{\partial \Pi^*} = \Pi + ieA_0\Phi^* \tag{3.9c}$$

$$\begin{aligned}-\partial_0 \Pi^* = \frac{\partial H_{\mathrm{T}}}{\partial \Phi^*} =& ie\Pi^* A_0 + e^2\big(A_1^2 + A_2^2\big)\Phi - \partial_1\partial_1\Phi - \partial_2\partial_2\Phi \\ &- ie(\partial_1\Phi)A_1 - ie(\partial_2\Phi)A_2 - ie\,\partial_1(A_1\Phi) \\ &- ie\,\partial_2(A_2\Phi) + \frac{\partial V}{\partial \Phi^*}\end{aligned} \tag{3.9d}$$

$$\partial_0 A_0 = \frac{\partial \mathcal{H}_{\mathrm{T}}}{\partial \Pi_0} = u \tag{3.9e}$$

$$-\partial_0 \Pi^0 = \frac{\partial H_{\mathrm{T}}}{\partial A_0} = -\partial_1 E_1 - \partial_2 E_2 + ie(\Pi^*\Phi^* - \Pi\Phi) \tag{3.9f}$$

$$\partial_0 A_1 = \frac{\partial H_{\mathrm{T}}}{\partial E_1} = e^2 E_1 + \partial_1 A_0 \tag{3.9g}$$

$$-\partial_0 E_1 = \frac{\partial H_{\mathrm{T}}}{\partial A_1} = 2e^2 A_1\Phi^*\Phi + ie(\Phi\,\partial_1\Phi^* - \Phi^*\,\partial_1\Phi) + \frac{1}{e^2}\partial_2 F_{12} \tag{3.9h}$$

$$\partial_0 A_2 = \frac{\partial H_{\mathrm{T}}}{\partial E_2} = e^2 E_2 + \partial_2 A_0 \tag{3.9i}$$

$$-\partial_0 E_2 = \frac{\partial H_{\mathrm{T}}}{\partial A_2} = 2e^2 A_2\Phi^*\Phi + ie(\Phi\,\partial_2\Phi^* - \Phi^*\,\partial_2\Phi) - \frac{1}{e^2}\partial_1 F_{12} \tag{3.9j}$$

$$\partial_0 u = \frac{\partial H_{\mathrm{T}}}{\partial \Pi_u} = 0 \tag{3.9k}$$

$$-\partial_0 \Pi_u = \frac{\partial H_{\mathrm{T}}}{\partial u} = \Pi_0 \tag{3.9l}$$

These are the equations of motion of the theory that preserve the constraints of the theory in the course of time. For the Poisson bracket $\{\ ,\ \}_{\rm p}$ of two functions A and B, we choose the following convention:

$$\{A(x), B(y)\}_{\rm p} := \int dz \sum_{\alpha} \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial P_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)} \right] \tag{3.10}$$

Demanding that the primary constraint χ_1 be preserved in the course of time, one obtains the secondary Gauss-law constraint of the theory as

$$\chi_2 := \{\chi_1, \mathcal{H}_{\rm T}\}_p = [\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^* \Phi^* - \Pi \Phi)] \approx 0 \tag{3.11}$$

The preservation of χ_2 for all times does not give rise to any further constraints. The theory is thus seen to possess only two constraints χ_1 and χ_2. Further, the matrix of the Poisson brackets of the constraints χ_1 is seen to be a null matrix, implying that the set of constraints χ_1 is first-class and that the theory described by (3.1) is a gauge-invariant theory. The action of the theory S(3.1) is in fact seen to be invariant under the time-dependent gauge transformations:

$$\delta\Phi = i\beta\Phi, \quad \delta\Phi^* = -i\beta\Phi^*, \qquad \delta u = -\partial_0\partial_0\beta \tag{3.12a}$$

$$\delta A_0 = -\partial_0\beta, \quad \delta A_1 = -\partial_1\beta, \qquad \delta A_2 = -\partial_2\beta \tag{3.12b}$$

$$\delta\Pi_u = \delta\Pi_0 = \delta E_1 = \delta E_2 = 0 \tag{3.12c}$$

$$\delta\Pi = -e\beta A_0\Phi^* - i\beta\, \partial_0\Phi^* + i(e-1)\Phi^*\, \partial_0\beta \tag{3.12d}$$

$$\delta\Pi^* = -e\beta A_0\Phi + i\beta\, \partial_0\Phi - i(e-1)\Phi\, \partial_0\beta \tag{3.12e}$$

where $\beta = \beta(t, x_1, x_2)$ is a function of the coordinates.

In order to quantize the theory using Dirac's procedure we convert the set of first-class constraints of the theory χ_i into a set of second-class constraints, by imposing, arbitrarily, some additional constraints on the system called gauge-fixing conditions or the gauge constraints. For this purpose, for the present theory, we could choose, for example, the set of gauge-fixing conditions: (A) $\rho_1 = A_0 = 0$ and $\rho_2 = A_1 = 0$; and (B) $\psi_1 = A_0 = 0$ and $\psi_2 = \partial_1 A_1 = 0$. Corresponding to these choices of the gauge-fixing conditions, we have the following two sets of constraints under which the quantization of the theory could be studied:

(A)

$$\xi_1 = \chi_1 = \Pi_0 \approx 0 \tag{3.13a}$$

$$\xi_2 = \chi_2 = [\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^*\Phi^* - \Pi\Phi)] \approx 0 \tag{3.13b}$$

$$\xi_3 = \rho_1 = A_0 \approx 0 \tag{3.13c}$$

$$\xi_4 = \rho_2 = A_1 \approx 0 \tag{3.13d}$$

and

(B)
$$\eta_1 = \chi_1 = \Pi_0 \approx 0 \tag{3.14a}$$
$$\eta_2 = \chi_2 = [\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^* \Phi^* - \Pi \Phi)] \approx 0 \tag{3.14b}$$
$$\eta_3 = \psi_1 = A_0 \approx 0 \tag{3.14c}$$
$$\eta_4 = \psi_2 = \partial_1 A_1 \approx 0 \tag{3.14d}$$

The matrices of the Poisson brackets among the set of constraints ξ_i and η_i are now seen to be nonsingular (and therefore inverible) and are omitted here for the sake of brevity.

The Dirac bracket $\{\ ,\ \}_{\rm D}$ of the two functions A and B is defined as (Dirac, 1950, 1964):

$$\{A, B\}_{\rm D} = \{A, B\}_{\rm p} - \iint dw\, dz \sum_{\alpha,\beta} \left[\{A, \Gamma_\alpha(w)\}_{\rm p} \right. \\ \left. \times \left[\Delta^{-1}_{\alpha\beta}(w, z)\right] \{\Gamma_\beta(z), B\}_{\rm p}\right] \tag{3.15}$$

where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(w, z)[=\{\Gamma_\alpha(w), \Gamma_\beta(z)\}_{\rm p}]$ is the matrix of the poisson brackets of the constraints Γ_i. The transition to quantum theory is made by the replacement of the Dirac brackets by the operator commutation relations according to

$$\{A, B\}_{\rm D} \to (-i)[A, B], \quad i = \sqrt{-1} \tag{3.16}$$

Finally, the nonvanishing equal-time commutators of the theory in Case A, i.e., in the gauge $A_0 = 0$ and $A_1 = 0$, are obtained as follows (Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995):

$$[\Phi(\vec{x}, t), \Pi(\vec{y}, t)] = i\delta(\vec{x} - \vec{y}) \tag{3.17a}$$
$$[\Phi^*(\vec{x}, t), \Pi^*(\vec{y}, t)] = i\delta(\vec{x} - \vec{y}) \tag{3.17b}$$
$$[A_2(\vec{x}, t), E_2(\vec{y}, t)] = i\delta(\vec{x} - \vec{y}) \tag{3.17c}$$
$$[A_1(\vec{x}, t), E_1(\vec{y}, t)] = 2i\delta(\vec{x} - \vec{y}) \tag{3.17d}$$
$$[\Phi(\vec{x}, t), E_1(\vec{y}, t)] = -\frac{1}{2} e\Phi\epsilon(x_1 - y_1)\delta(x_2 - y_2) \tag{3.17e}$$
$$[\Phi^*(\vec{x}, t), E_1(\vec{y}, t)] = \frac{1}{2} e\Phi^*\epsilon(x_1 - y_1)\delta(x_2 - y_2) \tag{3.17f}$$
$$[A_2(\vec{x}, t), E_1(\vec{y}, t)] = -\frac{1}{2} i\epsilon(x_1 - y_1)\delta'(x_2 - y_2) \tag{3.17g}$$

$$[\Pi(\vec{x},t),\,E_1(\vec{y},t)] = \frac{1}{2}e\Pi\epsilon(x_1-y_1)\delta(x_2-y_2) \tag{3.17h}$$

where $\epsilon(x_1-y_1)$ is a step function defined as

$$\epsilon(x_1-y_1) := \begin{cases} +1, & (x_1-y_1) > 0 \\ -1, & (x_1-y_1) < 0 \end{cases} \tag{3.18}$$

The nonvanishing equal-time commutators of the theory in Case B, i.e., in the gauge $A_0 = 0$ and $\partial_1 A_1 = 0$, are seen to be identical with those of Case A as they should, and are given by (3.17). This is not surprising in view of the fact that the gauges $A_1 = 0$ and $\partial_1 A_1 = 0$ conceptually mean the same thing.

For later use, for considering the BRST formulation of the theory we convert the total Hamiltonian density into the first-order Lagrangian density $\mathscr{L}_{10}$:

$$\begin{aligned}
\mathscr{L}_{10} :=\ & \Pi(\partial_0\Phi) + \Pi^*(\partial_0\Phi^*) + \Pi_0(\partial_0 A_0) + E_1(\partial_0 A_1) \\
& + E_2(\partial_0 A_2) + \Pi_u(\partial_0 u) - \mathcal{H}_T \\
=\ & \Pi(\partial_0\Phi) + \Pi^*(\partial_0\Phi^*) + E_1(\partial_0 A_1) + E_2(\partial_0 A_2) + \Pi_u(\partial_0 u) \\
& - \frac{1}{2}e^2\left(E_1^2 + E_2^2\right) - e^2\left(A_1^2 + A_2^2\right)\Phi^*\Phi - (\partial_1\Phi^*)(\partial_1\Phi) \\
& - (\partial_2\Phi^*)(\partial_2\Phi) - ie(\partial_1\Phi^*)A_1\Phi - ie(\partial_2\Phi^*)A_2\Phi \\
& + ieA_1\Phi^*(\partial_1\Phi) + ieA_2\Phi^*(\partial_2\Phi) - \frac{1}{2e^2}F_{12}^2 - V(|\Phi|^2)
\end{aligned} \tag{3.19}$$

In (3.19), the term $\Pi_0(\partial_0 A_0 - u)$ drops out in view of the Hamilton's equation (3.9e).

4. THE BRST FORMULATION

4.1. The BRST Invariance

For the BRST formulation of the Nielsen–Olesen model, we rewrite the theory as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of the gauge-invariant Nielsen–Olesen model and replace the notion of gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics, we then introduce new anticommuting variables c and $\bar{c}$ (Grassman numbers on the classical level and operators in the quantized theory) and a commuting variable b such that (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin, xxxx)

$$\hat{\delta}A_0 = \partial_0 c, \quad \hat{\delta}A_1 = -\partial_1 c, \quad \hat{\delta}A_2 = -\partial_2 c, \quad \hat{\delta}u = -\partial_0\partial_0 c, \quad \hat{\delta}\Phi = ic\Phi \quad (4.1a)$$

$$\hat{\delta}\Pi = [-ecA_0\Phi^* - ic\,\partial_0\Phi^* + i(e-1)\Phi^*\,\partial_0 c], \quad \hat{\delta}\Pi_u = 0, \quad \hat{\delta}\Phi^* = -ic\Phi^* \quad (4.1b)$$

$$\hat{\delta}c = 0, \qquad \hat{\delta}\overline{c} = b, \qquad \hat{\delta}b = 0 \quad (4.1c)$$

$$\hat{\delta}\Pi_0 = \hat{\delta}E_1 = \hat{\delta}E_2 = 0, \qquad \hat{\delta}\Pi^* = -ecA_0\Phi + ic\,\partial_0\Phi - i(e-1)\Phi\,\partial_0 c \quad (4.1d)$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function $f(A_0, A_1, A_2, u, \Phi, \Phi^*, c, \overline{c}, b, \Pi, \Pi^*, \Pi_0, \Pi_u, E_1, E_2, \Pi_c, \Pi_{\overline{c}}, \Pi_b)$ such that $\hat{\delta}f = 0$ (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin).

4.2. Gauge Fixing in the BRST Formalism

Performing gauge fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathscr{L}_{10}$, a trivial BRST-invariant function (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin, xxxx). We thus write the quantum Lagrangian density (taking, e.g., a trivial BRST-invariant function) as follows (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin):

$$\mathscr{L}_{\mathrm{BRST}} = \mathscr{L}_{10} - \hat{\delta}\left[\overline{c}\left(\partial_0 A_0 + \frac{1}{2}b\right)\right] \quad (4.2a)$$

$$\begin{aligned} &= \Pi(\partial_0\Phi) + \Pi^*(\partial_0\Phi^*) + E_1(\partial_0 A_1) + E_2(\partial_0 A_2) + \Pi_u(\partial_0 u) \\ &\quad - \frac{1}{2}e^2\left(E_1^2 + E_2^2\right) - e^2\left(A_1^2 A_2^2\right)\Phi^*\Phi - (\partial_1\Phi^*)(\partial_1\Phi) - (\partial_2\Phi^*)(\partial_2\Phi) \\ &\quad - ie(\partial_1\Phi^*)A_1\Phi - ie(\partial_2\Phi^*)A_2\Phi + ieA_1\Phi^*(\partial_1\Phi) \\ &\quad + ieA_2\Phi^*(\partial_2\Phi) - \frac{1}{2e^2}F_{12}^2 - V(|\Phi|^2) \\ &\quad + \hat{\delta}\left[\overline{c}\left(-\partial_0 A_0 - \frac{1}{2}b\right)\right] \end{aligned} \quad (4.2b)$$

The last term in the above equation (Eq. 4.2) is the extra BRST-invariant, gauge-fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathscr{L}_{\mathrm{BRST}}$ (with one integration by parts):

$$\begin{aligned}\mathscr{L}_{\text{BRST}} &= \Pi(\partial_0\Phi) + \Pi^*(\partial_0\Phi^*) + E_1(\partial_0 A_1) + E_2(\partial_0 A_2) + \Pi_u(\partial_0 u) \\ &\quad - \frac{1}{2}e^2\left(E_1^2 + E_2^2\right) - e^2\left(A_1^2 + A_2^2\right)\Phi^*\Phi - (\partial_1\Phi^*)(\partial_1\Phi) - (\partial_2\Phi^*)(\partial_2\Phi) \\ &\quad - ie(\partial_1\Phi^*)A_1\Phi - \frac{1}{2e^2}F_{12}^2 - ie(\partial_2\Phi^*)A_2\Phi + ieA_1\Phi^*(\partial_1\Phi) \\ &\quad + ieA_2\Phi^*(\partial_2\Phi) - V(|\Phi|^2) - \mathrm{b}(\partial_0 A_0) - \frac{1}{2}\mathrm{b}^2 + (\partial_0\overline{c})(\partial_0 c)\end{aligned} \tag{4.3}$$

Proceeding classically, the Euler–Lagrange equation for b reads (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin)

$$-b = \partial_0 A_0 \tag{4.4}$$

The requirement $\hat{\delta}b = 0$ (cf. 4.1c)) then implies

$$-\hat{\delta}b = \hat{\delta}(\partial_0 A_0) \tag{4.5}$$

which in turn implies

$$\partial_0(\partial_0 c) = 0 \tag{4.6}$$

The above equation is also an Euler–Langrange equation obtained by the variation of $\mathscr{L}_{\text{BRST}}$ with respect to $\overline{c}$. We now define the bosonic momenta in the usual way so that (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin)

$$\Pi_0 := \frac{\partial\mathscr{L}_{\text{BRST}}}{\partial(\partial_0 A_0)} = -\mathrm{b} \tag{4.7}$$

The fermionic momenta are, however, defined using the directional derivatives such that (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin)

$$\Pi_c := \mathscr{L}_{\text{BRST}}\frac{\overleftarrow{\partial}}{\delta(\partial_0 c)} = \partial_0\overline{c}; \qquad \Pi_{\overline{c}} := \frac{\overrightarrow{\partial}}{\delta(\partial_0\overline{c})}\mathscr{L}_{\text{BRST}} = \partial_0 c \tag{4.8}$$

implying that the variable canonically conjugate to c is $(\partial_0\overline{c})$ and the variable conjugate to $\overline{c}$ is $(\partial_0 c)$. In constructing the Hamiltonian density $\mathcal{H}_{\text{BRST}}$ from the Langrangian density in the usual way, one has to keep in mind that the former has to be Hermitian. Accordingly, we have

$$\begin{aligned}\mathcal{H}_{\mathrm{BRST}} &= \Pi_0(\partial_0 A_0) + \Pi^*(\partial_0\Phi^*) + \Pi(\partial_0\Phi) + \Pi_u(\partial_0 u) + E_1(\partial_0 A_1) + E_2(\partial_0 A_2)\\ &\quad + \Pi_c(\partial_0 c) + \Pi_{\bar{c}}(\partial_0\bar{c}) - \mathcal{L}_{\mathrm{BRST}}\\ &= \frac{1}{2}e^2\left(E_1^2 + E_2^2\right) + e^2\left(A_1^2 + A_2^2\right)\Phi^*\Phi + (\partial_1\Phi^*)(\partial_1\Phi) + (\partial_2\Phi^*)(\partial_2\Phi)\\ &\quad + ie(\partial_1\Phi^*)A_1\Phi + ie(\partial_2\Phi^*)A_2\Phi - ieA_1\Phi^*(\partial_1\Phi)\\ &\quad - ieA_2\Phi^*(\partial_2\Phi) + \frac{1}{2e^2}F_{12}^2 + V(|\Phi|^2) - \frac{1}{2}\Pi_0^2 + \Pi_c\Pi_{\bar{c}} \end{aligned} \tag{4.9}$$

We can check the consistency of (4.8) with (4.9) by looking at Hamilton's equations for the fermionic variables, i.e. (Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995)

$$\partial_0 c = \frac{\overrightarrow{\partial}}{\partial\Pi_c}\mathcal{H}_{\mathrm{BRST}}, \qquad \partial_0\bar{c} = \mathcal{H}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial\Pi_{\bar{c}}} \tag{4.10}$$

Thus we see that

$$\partial_0 c = \frac{\overrightarrow{\partial}}{\partial\Pi_c}\mathcal{H}_{\mathrm{BRST}} = \Pi_{\bar{c}}, \qquad \partial_0\bar{c} = \mathcal{H}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial\Pi_{\bar{c}}} = \Pi_c \tag{4.11}$$

is in agreement with (4.8). For the operators c, $\bar{c}$, $\partial_0 c$, and $\partial_0\bar{c}$, one needs to specify the anticommutation relations of $\partial_0 c$ with $\bar{c}$ or of $\partial_0\bar{c}$ with c, but not of c with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0, \qquad \partial_0\{\bar{c}, c\} = 0 \tag{4.12a}$$

$$\{\partial_0\bar{c}, c\} = -\{\partial_0 c, \bar{c}\} \tag{4.12b}$$

where { , } means an anticommutator. We thus see that the anticommutators in (4.12b) are nontrivial and need to be fixed. In order to fix these, we require that c satisfy the Heisenberg equation

$$[c, \mathcal{H}_{\mathrm{BRST}}] = i\,\partial_0 c \tag{4.13}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathcal{H}_{\mathrm{BRST}}] = \{\partial_0\bar{c}, c\}\,\partial_0 c \tag{4.14}$$

Equations (4.12)–(4.14) then imply

$$\{\partial_0\bar{c}, c\} = -\{\partial_0 c, \bar{c}\} = i \tag{4.15}$$

The minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory.

4.3. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformation (4.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators that satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy the following:

$$[\Phi, Q] = -iec\Phi, \qquad [A_0, Q] = \partial_0 c, \qquad [A_1, Q] = -\partial_1 c \tag{4.16a}$$

$$[\Pi, Q] = iec\Pi, \qquad [\Pi^*, Q] = -iec\Pi^* \tag{4.16b}$$

$$\{\bar{c}, Q\} = -\partial_1 c, \qquad [\Phi_0^*, Q] = iec\Phi^*, \qquad [A_2, Q] = -\partial_2 c \tag{4.16c}$$

$$\{\partial_0\bar{c}, Q\} = [ie(\Pi^*\Phi^* - \Pi\Phi) - \partial_1 E_1 - \partial_2 E_2] \tag{4.16d}$$

All other commutators and anticommutators involving Q vanish. In view of (4.16), the BRST charge operator for the present theory can be written as

$$Q = \int d^2x\, [ic\{\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^*\Phi^* - \Pi\Phi)\} - i(\partial_0 c)\Pi_0] \tag{4.17}$$

This equation implies that the set of states satisfying the condition

$$\Pi_0|\psi\rangle = 0 \tag{4.18a}$$

$$[\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^*\Phi^* - \Pi\Phi)]|\psi\rangle = 0 \tag{4.18b}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory, we write the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider Eq. (4.6) (namely, $\partial_0(\partial_0 c) = 0$). The solution of this equation gives the Heisenberg operator $c(t)$ (and correspondingly $\bar{c}\,(t)$) as (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a–c, 1994a–d, 1995; Nemeschansky *et al.*, 1988; Tyutin).

$$c(t) = Gt + F, \qquad \bar{c}(t) = G^\dagger t + F^\dagger \tag{4.19}$$

which at the time $t = 0$ imply

$$c \equiv c(0) = F, \qquad \bar{c} \equiv \bar{c}(0) = F^\dagger \tag{4.20a}$$

$$\partial_0 c \equiv \partial_0 c(0) = G, \qquad \partial_0\bar{c} \equiv \partial_0\bar{c}(0) = G^\dagger \tag{4.20b}$$

By imposing the conditions

$$c^2 = c^{-2} = \{\bar{c}, c\} = \{\partial_0\bar{c}, \partial_0 c\} = 0 \tag{4.21a}$$

$$\{\partial_0\bar{c}, c\} = i = -\{\partial_0 c, \bar{c}\} \tag{4.21b}$$

one then obtains

$$F^2 = F^{\dagger 2} = \{F^\dagger, F\} = \{G^\dagger, G\} = 0 \tag{4.22}$$

$$\{G^\dagger, F\} = -\{G, F^\dagger\} = i \tag{4.23}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$G|0\rangle = F|0\rangle = 0 \tag{4.24}$$

Defining $|0\rangle$ to have norm one, (4.23) implies

$$\langle 0|FG^\dagger|0\rangle = i, \qquad \langle 0|GF^\dagger|0\rangle = -i \tag{4.25}$$

so that

$$G^\dagger|0\rangle \neq 0, \qquad F^\dagger|0\rangle \neq 0 \tag{4.26}$$

The theory is thus seen to posses negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{\mathrm{BRST}}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators the Hamiltonian density is

$$\begin{aligned}\mathcal{H}_{\mathrm{BRST}} = {}& \frac{1}{2}e^2\big(E_1^2 + E_2^2\big) + e^2\big(A_1^2 + A_2^2\big)\Phi^*\Phi + (\partial_1\Phi^*)(\partial_1\Phi) + (\partial_2\Phi^*)(\partial_2\Phi) \\ & + ie(\partial_1\Phi^*)A_1\Phi + ie(\partial_2\Phi^*)A_2\Phi - ie(A_1\Phi^*)(\partial_1\Phi) - ie(A_2\Phi^*)(\partial_2\Phi) \\ & + \frac{1}{2e^2}F_{12}^2 + V(|\Phi|^2) - \frac{1}{2}\Pi_0^2 + G^\dagger G \end{aligned} \tag{4.27}$$

and the BRST charge operator Q is

$$Q = \int d^2x\,[iF\{\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^*\Phi^* - \Pi\Phi)\} - iG\Pi_0] \tag{4.28}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (4.18) holds, but also additional states for which

$$G|\psi\rangle = F|\psi\rangle = 0 \tag{4.29a}$$

$$\Pi|\psi\rangle \neq 0 \tag{4.29b}$$

$$[\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^*\Phi^* - \Pi\Phi)]|\psi\rangle \neq 0 \tag{4.29c}$$

The Hamiltonian is, however, also invariant under the anti-BRST transformations (in which the role of c and $-\bar{c}$ get interchanged) given by

$$\bar{\hat{\delta}}A_0 = \partial_0\bar{c}, \quad \bar{\hat{\delta}}A_1 = \partial_1\bar{c}, \quad \bar{\hat{\delta}}A_2 = \partial_2\bar{c}, \quad \bar{\hat{\delta}}\Phi = -i\bar{c}\Phi, \quad \bar{\hat{\delta}}\Phi^* = \quad i\bar{c}\Phi^*, \tag{4.30a}$$

$$\bar{\hat{\delta}}u = \partial_0\partial_0\bar{c}, \qquad \bar{\hat{\delta}}\Pi = e\bar{c}A_0\Phi^* + i\bar{c}\,\partial_0\Phi^* - i(e-1)\Phi^*\,\partial_0\bar{c}] \tag{4.30b}$$

$$\bar{\hat{\delta}}\Pi^* = ecA_0\Phi - i\bar{c}\,\partial_0\Phi + i(e-1)\Phi\,\partial_0\bar{c}, \qquad \bar{\hat{\delta}}\Pi_u = 0, \qquad \bar{\hat{\delta}}\Pi_0 = 0 \tag{4.30c}$$

$$\bar{\hat{\delta}}\bar{c} = 0, \qquad \bar{\hat{\delta}}c = -\mathrm{b}, \qquad \bar{\hat{\delta}}\mathrm{b} = 0, \qquad \bar{\hat{\delta}}E_1 = \bar{\hat{\delta}}E_2 = 0 \tag{4.30d}$$

with generator or anti-BRST charge

$$\bar{Q} = \int d^2x\,[-i\bar{c}\{\partial_1 E_1 + \partial_2 E_2 + ie(\Pi\Phi - \Pi^*\Phi^*)\} + i(\partial_0\bar{c})\Pi_0] \tag{4.31a}$$

$$= \int d^2x\,[-iF^\dagger\{\partial_1 E_1 + \partial_2 E_2 - ie(\Pi^*\Phi^* - \Pi\Phi)\} + iG^\dagger\Pi_0] \tag{4.31b}$$

we also have

$$[Q, H_{\mathrm{BRST}}] = [\bar{Q}, H_{\mathrm{BRST}}] = 0 \tag{4.32a}$$

$$H_{\mathrm{BRST}} = \int dx\,\mathcal{H}_{\mathrm{BRST}} \tag{4.32b}$$

and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that:

$$Q|\psi\rangle = 0 \tag{4.33a}$$

$$\bar{Q}|\psi\rangle = 0 \tag{4.33b}$$

The states for which (4.18) hold, satisfy both the above conditions (4.33a) and (4.33b), and in fact are the only states satisfying both of these conditions since, although with (4.22) and (4.23)

$$G^\dagger G = -GG^\dagger \tag{4.34}$$

there are no states of this operator with $G^\dagger|0\rangle = 0$ and $F^\dagger|0\rangle = 0$ (cf. Eq. (4.26)), and hence no free eigenstates of the fermionic part of H_{BRST} which are annihilated by each of G, $G^\dagger$, F, $F^\dagger$. Thus the only states satisfying (4.33) are those satisfying the constraints (3.5) and (3.11).

Further, the states for which (4.18) holds, satisfy both of the conditions (4.33a) and (4.33b), and in fact are the only states satisfying both of these conditions, because in view of (4.21), one cannot have simultaneously c, $\partial_0 c$, and $\bar{c}$, $\partial_0\bar{c}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (4.33) are those that satisfy the constraints of the theory [Eqs. (3.5) and (3.11)], and they belong to the set of BRST-invariant and anti-BRST-invariant states. One can understand the above point in terms of fermionic annihilation and creation operators as follows: The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (4.18) holds, but also additional states for which (4.29) holds. However, $\bar{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\bar{Q}$ contains only the states for which (4.18) holds, simply because $G^\dagger|\psi\rangle \neq 0$ and $F^\dagger|\psi\rangle \neq 0$. Thus, in this alternative way also we see that the states satisfying

$Q|\psi\rangle = \bar{Q}|\psi\rangle = 0$ (i.e., satisfying (4.33)) are only those that satisfy the constraints of the theory [Eqs. (3.5) and (3.11)] and also that these states belong to the set of BRST-invariant and anti-BRST-invariant states.

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